A new Osher Riemann solver for shallow water flow over fixed or mobile bed

D. Zugliani & G. Rosatti
Department of Civil, Environmental and Mechanical Engineering
University of Trento, Trento, Italy

ABSTRACT: The Osher solver (Osher & Solomon 1982) is a well-known numerical approach to estimate solutions of Riemann problems deriving from the finite volume method with Godunov fluxes applied to hyperbolic systems of Partial Differential Equations (PDEs). Recently, in Dumbser & Toro (2011b), the applicability of the solver has been extended to purely nonconservative systems. Nevertheless, shallow flows are described by a system where both conservative and non-conservative terms are present simultaneously. Some effort must be done to use the solver in this joined situation. In this work, we combined the conservative and non-conservative formulation ending up with a simple but powerful extension of the Osher solver suitable for the Shallow Water (SW) partially nonconservative PDEs systems. We also introduced a linear path in terms of primitive variables, instead of conserved ones. This approach reduces a little bit the computational cost in cases with simple linear relations between conserved and primitive variables (fixed-bed flows), while the cost reduction becomes more important when the relation is highly nonlinerar (mobile-bed flows) and the Jacobian of the fluxes can be expressed only in term of primitive variables. Finally, we exploited the possibility to use an explicit expression of the path integral of the non-conservative terms instead of a numerical approximation of it, e.g. the work of Rosatti & Begnudelli (2010).

1 INTRODUCTION

The 1D Shallow Water (SW) equations are the starting point for the description of several environmental phenomena like flash floods, hyperconcentrated flow, debris flow, etc. The Partial Differential Equations (PDEs) system is characterized by the presence of both conservative and nonconservative terms. Due to the hyperbolic nature of the PDEs system, a good numerical strategy for its integration is the finite volume method with Godunov type flux evaluation. Among many approximated Riemann solver present in the literature, a particular one is the Osher solver developed by Osher & Solomon (1982) and revisited by Dumbser & Toro (2011a) for the conservative and by Dumbser & Toro (2011b) for the nonconservative system from an 'universal' prospective (see section 2). In this paper (section 3) we introduce a new universal Osher solver that is a natural extension of original one. The numerical solver is applied (section 4) to the shallow water equation with discontinuous bed and to the two-phase isokinetic mobile bed equation.

2 EXISTING OSHER SOLVER

A generic one dimensional hyperbolic system composed by m partial differential equations (PDEs) conservation laws can be written in compact form as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}\left(\mathbf{U}\right) = 0 \qquad (1)$$

where $\mathbf{U}$ and $\mathbf{F}\left(\mathbf{U}\right)$ are the conserved variables and the conservative fluxes vector expressed in term of $\mathbf{U}$ respectively. The quasi-linear form of this system is

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}_U(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = 0 \qquad (2)$$

where $\mathbf{A}_U(\mathbf{U})$ is the Jacobian matrix of the conservative fluxes respect to the conserved variables

$$\mathbf{A}_U(\mathbf{U}) = \frac{\partial}{\partial \mathbf{U}}\mathbf{F}\left(\mathbf{U}\right) \qquad (3)$$

The numerical solution of equation (1) using the finite volume method reads

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}\right) \qquad (4)$$

where i states the cell $i\Delta x$, n is the time $n\Delta t$, $\mathbf{U}$ is the vector of the cell average conserved variables and $\mathbf{F}$ is the vector of the fluxes. The evaluation of the fluxes could be done using different approximated Riemann solver. One of them is the Osher solver developed by Osher & Solomon (1982) and it is based on the the assumption that it is possible to split the flux in a positive part and a negative one

$$\mathbf{F}(\mathbf{U}) = \mathbf{F}^{-}(\mathbf{U}) + \mathbf{F}^{+}(\mathbf{U}) \tag{5}$$

where

$$\frac{\partial \mathbf{F}^{-}}{\partial \mathbf{U}} = \mathbf{A}_U^{-} \qquad ; \qquad \frac{\partial \mathbf{F}^{+}}{\partial \mathbf{U}} = \mathbf{A}_U^{+} \tag{6}$$

Given the left value $\mathbf{U}_L$ and the right one $\mathbf{U}_R$ of any Riemann problem, the Osher flux is evaluated using the following expression

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}^{+}(\mathbf{U}_L) + \mathbf{F}^{-}(\mathbf{U}_R) \tag{7}$$

With some mathematical manipulation, it can be rewritten as

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \frac{1}{2}\int_{\mathbf{U}_L}^{\mathbf{U}_R} |\mathbf{A}_U| \, \mathrm{d}\mathbf{U} \tag{8}$$

where $\mathbf{F}_L$ and $\mathbf{F}_R$ are the fluxes evaluated using the left and the right conserved variables respectively, while

$$|\mathbf{A}_U| = \mathcal{R}|\mathbf{\Lambda}|\mathcal{R}^{-1} = \mathbf{A}_U^{+} - \mathbf{A}_U^{-} \tag{9}$$

and $\mathcal{R}$ is the matrix where the columns are the right eigenvectors of $\mathbf{A}_U$, $\mathcal{R}^{-1}$ is its inverse and $\mathbf{\Lambda}$ is the diagonal matrix of the eigenvalues.
In the expression (8) appears, as first term, the central part of the fluxes, while the integral represents the so called numerical viscosity.

Following Osher & Solomon (1982) it is necessary to know all the intermediate states of the solution (e.g. using rarefaction fan), in order to been able to solve the integral (8) in an explicit form. To overcome this difficulty, Dumbser & Toro (2011a) developed a new methodology by using an integral path in the phase-space. With this strategy, equation (8) can be rewritten as

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \frac{1}{2}\int_0^1 |\mathbf{A}_U(\mathbf{\Psi})| \frac{\partial \mathbf{\Psi}}{\partial s} \mathrm{d}s \tag{10}$$

where $\mathbf{\Psi}(s)$ is the path that links the left state $\mathbf{U}_L$ with the right one $\mathbf{U}_R$ in the phase-space. $\mathbf{\Psi}(s)$ is a Lipschitz continuous function defined in the interval $s \in [0, 1]$ with $\mathbf{\Psi}(0) = \mathbf{U}_L$ and $\mathbf{\Psi}(1) = \mathbf{U}_R$. The choice of the path is now independent of the intermediate states, making the solver easier to be implemented and used.

Several hyperbolic problems, however, are described by fully non-conservative systems. This type of systems can be written in vectorial form as

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{H}(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = 0 \tag{11}$$

where $\mathbf{H}(\mathbf{U})$ is the matrix of the non-conservative fluxes. These systems are formally different from the conservative ones (2) since $\mathbf{H}(\mathbf{U})$ is not the Jacobian of the fluxes. The solutions of these problems are quite different from the previous ones and need particular care in the development of a Riemann solver.
The extension of the Osher solver (10) to the non-conservative systems makes use of the theory of Dal Maso, LeFloch, & Murat (1995) on the non-conservative products. Following this theory the compatibility condition on the fluxes (for shake of clarity the fluxes for these type systems is represented with $\mathbf{G}$) can be written as

$$\mathbf{G}^{-}_{i+\frac{1}{2}} - \mathbf{G}^{+}_{i+\frac{1}{2}} = \int_0^1 \mathbf{H}(\mathbf{\Psi}) \frac{\partial \mathbf{\Psi}}{\partial s} \mathrm{d}s \tag{12}$$

and, from this definition, Dumbser & Toro (2011b) defined the fluxes for the Osher solver for fully non-conservative systems in the following way

$$\mathbf{G}^{+}_{i+\frac{1}{2}} = -\frac{1}{2}\int_0^1 (\mathbf{H}(\mathbf{\Psi}) + |\mathbf{H}(\mathbf{\Psi})|) \frac{\partial \mathbf{\Psi}}{\partial s} \mathrm{d}s \tag{13}$$

$$\mathbf{G}^{-}_{i+\frac{1}{2}} = \frac{1}{2}\int_0^1 (\mathbf{H}(\mathbf{\Psi}) - |\mathbf{H}(\mathbf{\Psi})|) \frac{\partial \mathbf{\Psi}}{\partial s} \mathrm{d}s \tag{14}$$

The numerical viscosity term that appears in expressions (10), (13) and (14) can be evaluated, following Dumbser & Toro (2011b), using the simplest path that connect the left and the right state. The proposed path is a straight-line segment defined as

$$\mathbf{\Psi}(\mathbf{U}_L, \mathbf{U}_R; s) = \mathbf{U}_L + s(\mathbf{U}_R - \mathbf{U}_L) \tag{15}$$

This path is used in many papers, among which we can cite Castro Díaz et al. (2013) where they say: "[...] *when there are no clear indication about the correct family of paths to be chosen, the family of straight segments is a sensible choice* [...]".

3 THROUGH A NEW OSHER SOLVER

The SW systems could not been correctly written by fully conservative (2) or non-conservative (11) PDEs, but they are described by equations composed with a mix of conservative and non-conservative terms. These mixed hyperbolic systems are written, in a quasi-liner form, as

$$\frac{\partial \mathbf{U}}{\partial t} + \mathcal{A}_U(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = 0 \tag{16}$$

where

$$\mathcal{A}_U(\mathbf{U}) = \mathbf{A}_U(\mathbf{U}) + \mathbf{H}(\mathbf{U}) \tag{17}$$

Equation (16) is similar to the quasi-linear form (2) of the conservative system but, in the present case, the matrix $\mathcal{A}(\mathbf{U})$ contains both the Jacobian of the conservative fluxes (as for the conservative case) and also the non-conservative fluxes (as for the non-conservative case). The finite volume method applied to this type of hyperbolic systems reads

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}^-_{i+\frac{1}{2}} - \mathbf{F}^+_{i-\frac{1}{2}}\right) \tag{18}$$

where, due to the non-conservative characteristics of some terms, the outgoing fluxes from the cell i (i.e. $\mathbf{F}^-_{i+\frac{1}{2}}$) are different from the incoming ones in the cell $i+1$ (i.e. $\mathbf{F}^+_{i+\frac{1}{2}}$).

In order to evaluate these fluxes, we assumes, in a similar way as for the original Osher solver, the existence of a splitting between the conservative fluxes and the non-conservative ones. In this way it is possible to write the following relation

$$\mathbf{F}^\pm_{i+\frac{1}{2}} = \mathbf{F}_{i+\frac{1}{2}} + \mathbf{G}^\pm_{i+\frac{1}{2}} \tag{19}$$

where $\mathbf{F}_{i+\frac{1}{2}}$ is described by equation (10) and $\mathbf{D}^\pm_{i+\frac{1}{2}}$ with the equations (13) and (14). With this statement we can write

$$\mathbf{F}^\pm_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \frac{1}{2}\int_0^1 |\mathcal{A}_U| \frac{\partial \mathbf{\Psi}}{\partial s}\, \mathrm{d}s - \left(\pm\frac{1}{2}\int_0^1 \mathbf{H}\frac{\partial \mathbf{\Psi}}{\partial s}\, \mathrm{d}s\right) \tag{20}$$

that is the New Osher solver in Conservative variables (NOC) for a generic hyperbolic system containing conservative and non-conservative terms. From now on, for shake of clarity, we neglect the dependency of $\mathbf{\Psi}$ in the matrices.

We can easily check that for a conservative system (i.e. $\mathbf{H} = 0$ and $|\mathcal{A}_U| = |\mathbf{A}_U|$), the solver reduces exactly the conservative one (10). Also the compatibility condition that reads

$$\mathbf{F}^-_{i+\frac{1}{2}} - \mathbf{F}^+_{i+\frac{1}{2}} = \int_0^1 \mathbf{H}\frac{\partial \mathbf{\Psi}}{\partial s}\, \mathrm{d}s \tag{21}$$

as for the non-conservative system could be easily proofed.

3.1 *New Osher solver in primitive variables*

This new Osher solver in conservative variables is developed for an hyperbolic system of PDEs where the fluxes could be written using the conservative variables. An example is the well known Shallow Water Equations with discontinuous bed (SWE) analyzed later on. Nevertheless, writing the fluxes for an hyperbolic system in term of conservative variables is not always possible. Indeed, one example is the system of PDEs describing the two-phase isokinetic flow over mobile bed (see section 4.2). However is possible to rewrite these type of systems in compact form as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{W}) + \mathbf{H}(\mathbf{W})\frac{\partial \mathbf{W}}{\partial x} = 0 \tag{22}$$

where $\mathbf{U}$ and $\mathbf{W}$ are the vectors of the conserved and primitive variables respectively, while $\mathbf{F}(\mathbf{W})$ and $\mathbf{H}(\mathbf{W})$ are the conservative and non-conservative fluxes expressed in term of primitive variables.
In quasi linear form, this type of PDEs system reads

$$\frac{\partial \mathbf{U}}{\partial t} + \mathcal{A}_W\frac{\partial \mathbf{U}}{\partial x} = 0 \tag{23}$$

where

$$\mathcal{A}_W = (\mathbf{A}_W(\mathbf{W}) + \mathbf{H}(\mathbf{W}))\,\mathbf{B}(\mathbf{W})^{-1} \tag{24}$$

and

$$\mathbf{A}_W(\mathbf{W}) = \frac{\partial \mathbf{F}(\mathbf{W})}{\partial \mathbf{W}} \tag{25}$$

is the Jacobian matrix of the conservative fluxes respect to the primitive variables, while

$$\mathbf{B}(\mathbf{W}) = \frac{\partial \mathbf{U}}{\partial \mathbf{W}} \tag{26}$$

is the inverse of the Jacobian of the conserved variables respect to the primitive ones.

Comparing system (16) and (23) we notice that they are similar except for the definition of the matrix $\mathcal{A}$, so the key idea is to use the NOC solver (20) changing the matrices in the two integrals. This lead to the following expression for the fluxes

$$\mathbf{F}^\pm_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \frac{1}{2}\int_0^1 |\mathcal{A}_W| \frac{\partial \mathbf{\Psi}}{\partial s}\, \mathrm{d}s - \left(\pm\frac{1}{2}\int_0^1 \mathbf{H}\mathbf{B}^{-1}\frac{\partial \mathbf{\Psi}}{\partial s}\, \mathrm{d}s\right) \tag{27}$$

This simple extension has the disadvantage that the path integration is defined in conserved variables, while the matrices $\mathbf{A}_W$, $\mathbf{B}$ and $\mathbf{H}$ depends on the primitive ones therefore, for the evaluation of the two integrals, a change of variable from the reconstructed conserved variables to the reconstructed primitive ones is necessary. For many systems, e.g. the mobile bed system, the passage from conserved to primitive variables is quite complicated and leads to a non-liner system that must be solved several time during the integration process since, as we explained later on in section 3.3, it is done in a numerical way.

Our idea is to change the path integration from the conserved variables to the primitive ones in order to overcome the complexity related to the solution of the non-linear system. Omitting all the details for the derivations, we end up with a New Osher solver with Primitive variable (NOP) that reads

$$\mathbf{F}^{\pm}_{i+\frac{1}{2}} = \frac{1}{2}\left(\mathbf{F}_L + \mathbf{F}_R\right) - \left(\pm\frac{1}{2}\int_0^1 \mathbf{H}\frac{\partial \mathbf{\Psi}_P}{\partial s}\,\mathrm{d}s\right) - \frac{1}{2}\int_0^1 |\mathcal{A}_W|\,\mathbf{B}\frac{\partial \mathbf{\Psi}_P}{\partial s}\,\mathrm{d}s \quad (28)$$

where $\mathbf{\Psi}_P\left(s;\mathbf{W}_L,\mathbf{W}_R\right)$ is the path in primitive variable connecting the left values $\mathbf{W}_L$ and the right ones $\mathbf{W}_R$. Since the path in primitive variables must have all the property of the path in conserved variables and following the approach used in Dumbser & Toro (2011b) it is possible to use the segment linear path also in the NOP in order to maintain the generality of the scheme. This linear path in primitive variables reads

$$\mathbf{\Psi}_P\left(\mathbf{W}_L,\mathbf{W}_R;s\right) = \mathbf{W}_L + s\left(\mathbf{W}_R - \mathbf{W}_L\right) \quad (29)$$

and thanks to it we can write the general structure for the NOP

$$\mathbf{F}^{\pm}_{i+\frac{1}{2}} = \frac{1}{2}\left(\mathbf{F}_L + \mathbf{F}_R\right) - \frac{1}{2}\left(\int_0^1 |\mathcal{A}_W|\,\mathbf{B}\,\mathrm{d}s + \left(\pm\int_0^1 \mathbf{H}\,\mathrm{d}s\right)\right)\left(\mathbf{W}_R - \mathbf{W}_L\right) \quad (30)$$

where the first term is the central part of the fluxes, the first integral represents the so called numerical viscosity that derives from the conservative and non-conservative fluxes, while the second integral is the integral value of the non-conservative fluxes across the interface of the computational cell.

3.2 *Discretization of the non-conservative terms*

For the free surface flow, the non-conservative terms has an important role in the solution of the Riemann problems, so particular attention must be paid for its discretization. The physical meaning of the integral of the non-conservative term, i.e. the second integral in equation (28), can be analyzed referring to the theory developed by Dal Maso et al. (1995) about the non-conservative product. Following this theory, the weak solution of (23) across a discontinuity must satisfy

$$\int_0^1 \left(-S_S\mathbf{B} + \mathbf{A}_W + \mathbf{H}\right)\frac{\partial \mathbf{\Psi}_P}{\partial s}\,\mathrm{d}s = 0 \quad (31)$$

where S_S is the speed of the traveling discontinuity. Using the definition (25) and (26) it is possible to rewrite this expression as

$$S_S\left(\mathbf{U}_R - \mathbf{U}_L\right) = \mathbf{F}_R - \mathbf{F}_L + \int_0^1 \mathbf{H}\frac{\partial \mathbf{\Psi}_P}{\partial s}\,\mathrm{d}s \quad (32)$$

where last term of the equation is the integral we are referring to, while the other terms are the classical Rankine Hugoniot (RH) condition. As proved in Rosatti & Begnudelli (2010) for the standing contact wave of the SWE with discontinuous bed and in Rosatti & Fraccarollo (2006) for the shock waves of the two phase mobile bed, the only relation valid for the description of this type of waves is the Generalized Rankine Hugoniot (GRH) conditions that reads

$$S_S\left(\mathbf{U}_R - \mathbf{U}_L\right) = \mathbf{F}_R - \mathbf{F}_L - \mathbf{D} \quad (33)$$

where $\mathbf{F}_L$ and $\mathbf{F}_R$ is the conservative fluxes on the left and right of the wave, while $\mathbf{D}$ represents the pressure exerted by the fluid on the bottom step.
Comparing equation (33) with (32) we notice that, for the free surface flow, the integral of the non-conservative term is nothing more than the pressure exerted by the fluid on the bed step

$$\int_0^1 \mathbf{H}\frac{\partial \mathbf{\Psi}_P}{\partial s}\,\mathrm{d}s = -\mathbf{D} \quad (34)$$

so, in the integration of the non-conservative term in the NOP solver (30) it is necessary to use the correct path in order to satisfy equation (34). Since a general expression for the path that corroborates this equation is not easy to find or is quite complicated to deal with (e.g. the path proposed by Cozzolino et al. (2011)) our cutting-edge idea is the introduction in the NOP solver of the exact integral of the non-conservative terms so equation (34).
Since we want to maintain also a part of the universality of the solver, we can use the linear path in the numerical viscosity term since no constrains on its are provided. The final version of the NOP, proposed in this article, is therefore

$$\mathbf{F}^{\pm}_{i+\frac{1}{2}} = \frac{1}{2}\left(\mathbf{F}_L + \mathbf{F}_R\right) - \left(\pm\frac{1}{2}\mathbf{D}\right) - \frac{1}{2}\left(\int_0^1 |\mathcal{A}_W|\,\mathbf{B}\,\mathrm{d}s\right)\left(\mathbf{W}_R - \mathbf{W}_L\right) \quad (35)$$

3.3 *Numerical discretization*

In the final form of the proposed NOP solver (35), the numerical viscosity term must be evaluated via the solution of the integral itself. Since a closed form of this integral is not always available, it is possible to solve it using a numerical method (e.g. midpint rule, trapezoidal rule, ...). Among these methods, a good accuracy can be obtained using the Gauss-Legendre (GL) quadrature rule with three points. Following this method the integral is replaced with a summation and the resulting NOP solver is

$$\mathbf{F}^{\pm}_{i+\frac{1}{2}} = \frac{1}{2}\left(\mathbf{F}_L + \mathbf{F}_R\right) - \left(\pm\frac{1}{2}\mathbf{D}\right) - \frac{1}{2}\left(\sum_{i=1}^{3}\omega_i \left|\mathcal{A}_W\right| \mathbf{B}\right)\left(\mathbf{W}_R - \mathbf{W}_L\right) \quad (36)$$

where s_i is the position and ω_i is the weight. For the three point GL rule in the integration domain $s \in [0;1]$, the positions and weights read

$$s_{1,3} = \frac{1}{2} \pm \frac{\sqrt{15}}{10}, \; s_2 = \frac{1}{2} \quad (37)$$

$$\omega_{1,3} = \frac{5}{18}, \; \omega_2 = \frac{8}{18} \quad (38)$$

The use of GL rule for the integral of the numerical viscosity produce a simple, powerful and physically based Osher Riemann solver where the only information needed are the eigenstructure of the hyperbolic problem.

4 TEST CASES

In this section we present two applications of the new Osher solver (both NOC and NOP) presented in this paper. In particular we apply the Riemann solver to the SWE with discontinuous bed in Section 4.1, while in Section 4.2 we solve the two-phase isokinetic mobile bed system.

4.1 *Fixed bed system*

The homogeneous part of the classical SWE with discontinuous bed is

$$\begin{cases} \dfrac{\partial}{\partial t}\left(h + z_b\right) + \dfrac{\partial}{\partial x}uh = 0 \\ \dfrac{\partial}{\partial t}uh + \dfrac{\partial}{\partial x}\left(u^2 h + g\dfrac{h^2}{2}\right) + gh\dfrac{\partial}{\partial x}z_b = 0 \\ \dfrac{\partial}{\partial t}z_b = 0 \end{cases} \quad (39)$$

where $h$is the water depth, u is the water velocity, g in the constant gravity acceleration, z_b is the bed elevation, x is the longitudinal direction and t is the time as sketched in figure 1. For all the details on this type of system, we refer to the works of LeFloch & Thanh (2007) and Rosatti & Begnudelli (2010).

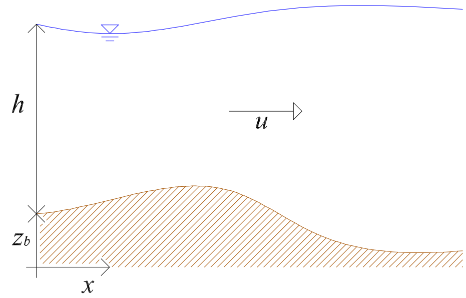

Figure 1: Sketch of the variables for the fixed bed case.

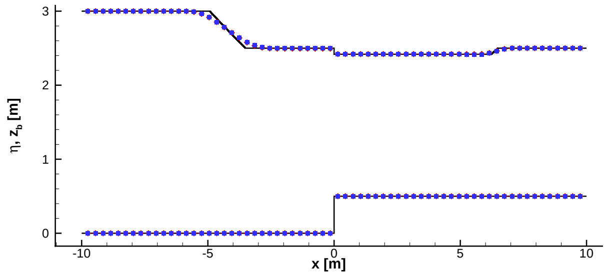

Figure 2: Comparison between analytic solution (black line) and numerical solution with NOP solver without (red dots) and with (blue dots) correction on the non-conservative terms discretization for the fixed bed Riemann problem test case. In the picture are represented the free surface $\eta = h + z_b$ and the bottom elevation z_b.

The test case proposed is the solution of a Riemann problem where the left state is $h_L = 3.0m$, $u_L = 0.5m/s$ and $z_{bL} = 0.0m$, while the right state is define by $h_R = 2.0m$, $u_R = 2.07m/s$ and $z_{bR} = 0.5m$. The domain is described with 250 cells, the final integration time step is $1.0s$ and the Courant number is set equal to 0.9.

In figure 2 is represented the comparison between analytical solution (in black solid line) and the numerical result obtained using the NOP solver without (in red dots) and with (in blue dots) corrected non-conservative term. At a first glance the two numerical solutions agree well with the analytical one. However, zooming in at the free surface, near the bed discontinuity (figure 3), we highlight that only the Osher solver with the correct discretizatio of the non-conservative terms reproduces in a satisfactory way the exact solution.

4.2 *Mobile bed system*

The system of PDEs describing the two-phase isokinetic flow over mobile bed (see Armanini et al. (2009), Cao et al. (2006) and Murillo & García-Navarro (2010) for all the details on this type of sys-

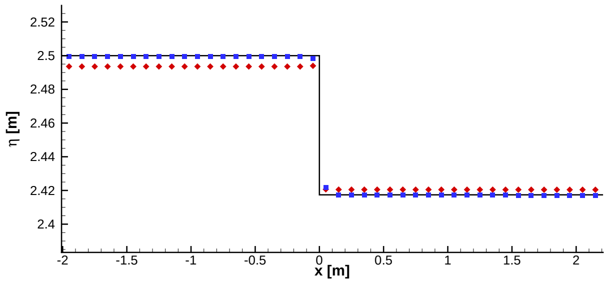

Figure 3: Zoom of the free surface elevation near the bed step for the fixed bed Riemann problem test case. Black line is the exact solution while red dots and blue dots represents the NOP solver without and with correct discretization for the non-conservative term.

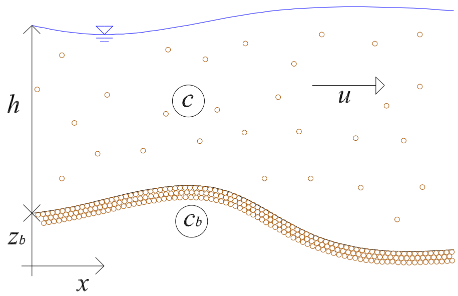

Figure 4: Sketch of the variables for the mobile bed case.

tem) is

$$\begin{cases} \frac{\partial}{\partial t}(h+z_b)+\frac{\partial}{\partial x}uh=0 \\ \frac{\partial}{\partial t}(1+c\Delta)\,uh+gh\,(1+c\Delta)\,\frac{\partial}{\partial x}z_b+ \\ \qquad +\frac{\partial}{\partial x}(1+c\Delta)\left(u^2h+g\frac{h^2}{2}\right)=0 \\ \frac{\partial}{\partial t}(c_bz_b+ch)+\frac{\partial}{\partial x}cuh=0 \end{cases} \tag{40}$$

where h is the flow depth, z_b is the bottom elevation, u is the velocity, Δ is the constant submerged relative density for the solid phase, g in the constant gravity acceleration, x is the longitudinal direction , t is the time, c is the concentration of the solid phase in the flow and c_b is the constant bottom concentration as sketched in figure 4. In order to solve this PDEs system it is necessary to introduce an algebraic closure formulation relating the concentration with the local hydrodynamic values, like

$$c=\text{func}\,(u,h) \tag{41}$$

Many formulations can be found in the literature (we refer to the work of Wu (2007) for a possible list of them) and here we use the simple relation proposed by Armanini et al. (2009)

$$c=\beta\frac{u^2}{gh} \tag{42}$$

where β is a constant dimensionless transport parameter.

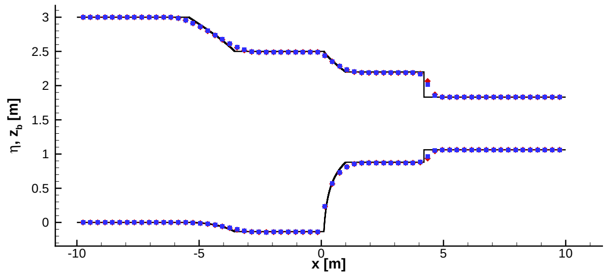

Figure 5: Comparison between analytic solution (black line) and numerical solution with NOP solver without (red dots) and with (blue dots) correction on the non-conservative terms discretization for the mobile bed Riemann problem test case. In the picture are represented the free surface $\eta=h+z_b$ and the bottom elevation z_b.

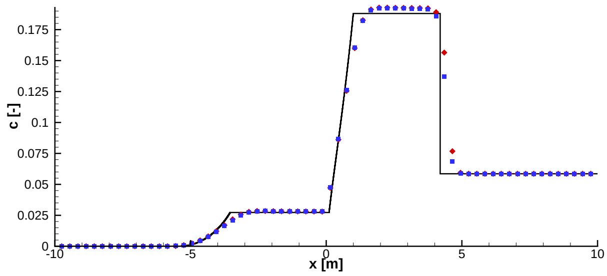

Figure 6: Comparison between analytic solution (black line) and numerical solution with NOP solver without (red dots) and with (blue dots) correction on the non-conservative terms discretization for the mobile bed Riemann problem test case. In the picture is represented the concentration c.

The test case proposed is the solution of a Riemann problem where the left state is $h_L=3.0m$, $u_L=0.0m/s$ and $z_{bL}=0.0m$, while the right state is define by $h_R=0.770m$, $u_R=0.665m/s$ and $z_{bR}=1.06m$. The other parameter used are $\beta=1.0$, $c_b=0.65$ and $\Delta=1.65$. The domain is described with 250 cells, the final integration time step is $1.0s$ and the Courant number is set equal to 0.9.
In figures 5 and 6 are represented the comparison between analytical solution (in black solid line) and the numerical results obtained using the NOP solver without (in red dots) and with (in blue dots) corrected non-conservative term. At a first glance the two numerical solutions agree well with the analytical one. However, zooming in at the free surface near the shock wave (figure 7), we highlight that the Osher solver with the correct discretizatio of the non-conservative terms reproduces in a better way the exact solution.

We perform lots of other simulation both with the new Osher solver in primitive variables and the one with the conserved variables in order to evaluate their time consumption. With a systematic comparison between the two solvers, we highlight a speed up of about 3.3% for the NOP respect to the NOC. This is an advantage for the NOP solver respect to the NOC one, since saving 3.3% of time in a simulation of a realistic free surface flow event, where the timing is of the order of hours, allows to obtain accurate solutions a little bit faster.

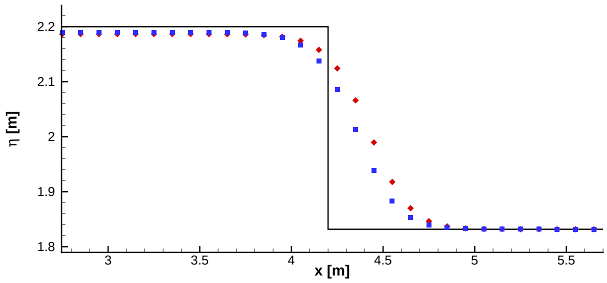

Figure 7: Zoom of the free surface elevation for the mobile bed Riemann problem test case near the shock wave. Black line is the exact solution while red dots and blue dots represents the NOP solver without and with correct discretization for the non-conservative term.

5 CONCLUSIONS

With this work, we developed a powerful but simple extension of the Osher solver for partially non-conservative PDEs systems, such as the systems describing free surface flow, using a linear path integration for the numerical viscosity and an exact integration for the non-conservative terms. The path integration is performed using both conserved and primitive variable, and the two obtained solvers (respectively NOC and NOP) are applied to the well known SWE with bed discontinuity and two-phase isokinetic mobile bed equations. With some test cases we have highlighted that only using the corrected discretization of the non-conservative term, the numerical solution have a good agreement with the analytical one. We also highlight that the NOP solver is 3.3% faster than the NOC one when the Jacobian of the fluxes can be expressed only in term of primitive variables.
The extension to a 2D model of the new Osher solver will be done in a forthcoming works.

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LIST OF SYMBOLS

c	concentration
c_b	bottom concentration
g	gravity acceleration
h	water depth
i	cell index
n	time index
s	integration variable
t	time
u	flow velocity
x	space variable
z_b	bottom elevation
$\mathbf{A}_X$	Jacobian matrix of the fluxes respect to the generic vector $\mathbf{X}$
$\mathbf{B}$	Jacobian matrix of the conserved variables respect to the primitive ones
$\mathbf{D}$	pressure exerted by the fluid on the bed step
$\mathbf{F}$	vector of the fluxes
$\mathbf{G}$	vector of the fluxes for non-conservative case
$\mathbf{H}$	matrix of non-conservative fluxes
S_s	shock speed
$\mathbf{U}$	vector of conserved variables
$\mathbf{W}$	vector of primitive variables

β	dimensionless transport parameter
η	free surface elevation
ω	Gauss-Legendre weight

Δ	submerged relative density for solid phase
Δt	time step integration
Δx	space discretization
$\mathbf{\Lambda}$	diagonal matrix of eigenvalues
$\mathbf{\Psi}$	path in conserved variables
$\mathbf{\Psi}_P$	path in primitive variables

$\mathcal{R}$	matrix of the right eigenvectors

$\mathbf{X}_{L,R}$	generic vector $\mathbf{X}$ evaluated on the left or right of a discontinuity
$\mathbf{X}_{i+\frac{1}{2}}$	generic vector $\mathbf{X}$ evaluated on the interface $i+\frac{1}{2}$

REFERENCES

Armanini, A., L. Fraccarollo, & G. Rosatti (2009). Two-dimensional simulation of debris flows in erodible channels. *Computers & Geosciences 35*(5), 993–1006.

Cao, Z., G. Pender, & P. A. Carling (2006). Shallow water hydrodynamic models for hyperconcentrated sediment-laden floods over erodible bed. *Advances in W 29*(4), 546–557.

Castro Díaz, M. J., E. D. Fernández-Nieto, T. Morales de Luna, G. Narbona-Reina, & C. Parés (2013). A HLLC scheme for nonconservative hyperbolic problems. Application to turbidity currents with sediment transport. *ESAIM: Mathematical Modelling and Numerical Analysis 47*(1), 1–32.

Cozzolino, L., R. Della Morte, C. Covelli, G. Del Giudice, & D. Pianese (2011). Numerical solution of the discontinuous-bottom Shallow-water Equations with hydrostatic pressure distribution at the step. *Advances in Water Resources 34*(11), 1413–1426.

Dal Maso, G., P. G. LeFloch, & F. Murat (1995). Definition and weak stability of nonconservative products. *Journal de Mathématiques Pures et Appliquées 74*(6), 483–548.

Dumbser, M. & E. F. Toro (2011a). On universal Osher type schemes for general nonlinear hyperbolic conservation laws. *Communications in Computational Physics 10*(3), 635–671.

Dumbser, M. & E. F. Toro (2011b). A simple extension of the Osher Riemann solver to non-conservative hyperbolic systems. *Journal of Scientific Computing 48*(1-3), 70–88.

LeFloch, P. G. & M. D. Thanh (2007). The Riemann problem for the shallow water equations with discontinuous topography. *Communications in Mathematical Sciences 5*(4), 865–885.

Murillo, J. M. & P. García-Navarro (2010). An Exner-based coupled model for two-dimensional transient flow over erodible bed. *Journal of Computational Physics 229*(23), 8704–8732.

Osher, S. & F. Solomon (1982). Upwind difference schemes for hyperbolic system of conservation laws. *Mathematics of Computation 38*(158), 339–374.

Rosatti, G. & L. Begnudelli (2010). The Riemann problem for the one-dimensional, free-surface shallow water equations with a bed step: Theoretical analysis and numerical simulations. *Journal of Computational Physics 229*(3), 760–787.

Rosatti, G. & L. Fraccarollo (2006). A well-balanced approach for flows over mobile-bed with high sediment-transport. *Journal of Computational Physics 220*(1), 312–338.

Wu, W. (2007). *Computational River Dynamics*. Taylor & Francis.